arXiv:1304.4290v1 [math.AC] 15 Apr 2013

CLASSES OF COMPLETE INTERSECTION NUMERICAL SEMIGROUPS

MARCO D'ANNA, VINCENZO MICALE AND ALESSIO SAMMARTANO

Abstract. We consider several classes of complete intersection numerical semigroups, namely telescopic and free semigroups, semigroups associated to plane branches, semigroups with a unique Betti element and those with α-rectangular, β-rectangular and γ-rectangular Apéry set. We study all the logical implications among these classes and provide examples. Most of these classes are shown to be well-behaved with respect to the operation of gluing.

Introduction

The concept of complete intersection is one of the most prominent in algebraic geometry. The notion of complete intersection for numerical semigroups (i.e. submonoids of $(\mathbb{N},+)$) was introduced by Herzog in [19], where he proved the celebrated theorem stating that a three-generated semigroup is a complete intersection if and only if it is symmetric. Complete intersection semigroups have been studied extensively since then (see e.g. [2], [7], [8], [16], [26], [28]).

Several subclasses of the complete intersections have been investigated, with different motivations arising from algebra and geometry. The study of the value-semigroup of plane algebroid branches was initiated by Apéry in his famous paper [1] and then continued by several other authors (e.g. [4], [10], [29]). Bertin and Carbonne defined free numerical semigroups in [6] in order to generalize a formula for the conductor of the local ring of a plane branch in terms of its Puiseux expansion. Telescopic semigroups were introduced in [21] for their applications to codes, but they have also been studied in connection with homology (cf. [22]) and factorization theory (cf. [27]). Numerical semigroups with β-rectangular and γ-rectangular Apéry set were defined in [14] to characterize semigroup rings whose tangent cone is a complete intersection. Finally, semigroups having a unique Betti element were characterized in [18].

The main purpose of this paper is to understand better the classes mentioned above and the relations among them. We also introduce a new class which is naturally related to the previous ones, semigroups with α-rectangular Apéry set. Our main result is Theorem 1.13 in which we show that the implications in Figure 1 hold and provide counterexamples for the "missing arrows". Some of these implications are somewhat surprising: despite the fact that the definitions of free and telescopic semigroups are very similar, two classes of semigroups with rectangular Apéry sets sit between them. In Section 2 we study the operation of gluing, which allows to produce new complete intersection semigroups from old ones. We show that semigroups with α-rectangular Apéry sets are also, in some sense, well-behaved with respect to gluing. We conclude with some applications to known results in literature.

2000 *Mathematics Subject Classification.* 14M10; 20M14.

Key words and phrases. Numerical semigroup; complete intersection; rectangular Apéry set; telescopic semigroup; free semigroup; semigroup of a plane branch; gluing.

$$
\begin{tikzcd}
\text{Plane branch} \arrow[d, Rightarrow] \arrow[drr, Rightarrow] & & \text{Unique Betti element} \arrow[d, Rightarrow] \arrow[dll, Rightarrow] \\
\text{Telescopic} \arrow[dr, Rightarrow] & & \alpha\text{-rectangular Ap}(S) \arrow[dl, Rightarrow] \\
& \beta\text{-rectangular Ap}(S) \arrow[d, Rightarrow] & \\
& \gamma\text{-rectangular Ap}(S) \arrow[d, Rightarrow] & \\
& \text{Free} \arrow[d, Rightarrow] & \\
& \text{Complete intersection} &
\end{tikzcd}
$$

FIGURE 1. Logical implications in Theorem 1.13.

Computations were performed by using GAP (cf. [15],[17]). The tests for the properties treated in this paper will be included in the next release of the package `NumericalSgps`.

1. THE CLASSES

We start by giving some preliminaries on numerical semigroups. Let $\mathbb{N}$ denote the set of non-negative integers. A **numerical semigroup** is a subset $S \subseteq \mathbb{N}$ that is closed under addition, contains 0 and has finite complement in $\mathbb{N}$. The largest integer in $\mathbb{Z} \setminus S$ is called **Frobenius number** of S and is denoted by $f = f(S)$, whereas the smallest positive integer in S is known as **multiplicity** of S and is denoted by $m = m(S)$.

We define a partial order on S setting $s \preceq t$ if there is an element $u \in S$ such that $t = u + s$. The set of minimal elements in the poset $(S \setminus \{0\}, \preceq)$ is called **minimal system of generators** of S: all integers in S are linear combinations of minimal elements with coefficients in $\mathbb{N}$. We define the **embedding dimension** of S as the cardinality of its minimal system of generators and denote it by $\nu = \nu(S)$; it is easy to see that $\nu(S) \leq m(S)$. A numerical semigroup minimally generated by $\{g_1, \dots, g_\nu\}$ will be denoted by $\langle g_1, \dots, g_\nu \rangle$. The condition $|\mathbb{N} \setminus S| < \infty$ is equivalent to $\gcd(g_1, \dots, g_\nu) = 1$.

For any $n \in S$ we define the **Apéry set of S with respect to n** as $\mathrm{Ap}(S, n) = \{s \in S \mid s - n \notin S\}$, or equivalently $\mathrm{Ap}(S, n) = \{\omega_0, \dots, \omega_{n-1}\}$ where $\omega_i = \min\{s \in S : s \equiv i \pmod{n}\}$. The smallest element in $\mathrm{Ap}(S, n)$ is 0, while the largest one is $f(S) + n$. If

$n = m(S)$ is the multiplicity we just write $\mathrm{Ap}(S)$ in place of $\mathrm{Ap}(S,n)$, and we will refer to it simply as the Apéry set of S.

Two types of semigroups are among the most studied, mainly for their relevance in algebraic geometry. A semigroup S is called **symmetric** if, for any $x \in \mathbb{Z}$, we have $x \in S \Leftrightarrow f(S) - x \notin S$; this condition is equivalent to the fact that $f(S) + m(S)$ is the unique maximal element of the poset $(\mathrm{Ap}(S), \preceq)$. A semigroup S is called a **complete intersection** if the semigroup ring $\Bbbk[[t^S]]$ is complete intersection, or equivalently if the cardinality of any of its minimal presentations equals $\nu(S) - 1$ (cf. [25], page 129).

Numerical semigroups other than $\mathbb{N}$ are never unique factorization monoids, as there are always elements with different decompositions into irreducibles (note that in our context an irreducible element is the same thing as a minimal generator). If $s = \lambda_1 g_1 + \cdots + \lambda_\nu g_\nu$ with $\lambda_i \in \mathbb{N}$ we say that $\lambda_1 g_1 + \cdots + \lambda_\nu g_\nu$ is a **representation** of s. Given $s \in S$, we define the M**-adic order** as $\mathrm{ord}(s) = \max\{\sum_{i=1}^{\nu} \lambda_i \mid \sum_{i=1}^{\nu} \lambda_i g_i \text{ is a representation of } s\}$. We say that $s = \lambda_1 g_1 + \cdots + \lambda_\nu g_\nu$ is a **maximal representation** of s if $\sum_{i=1}^{\nu} \lambda_i = \mathrm{ord}(s)$. We can define an other partial order on S setting $s \preceq_M t$ if there exists $u \in S$ such that $s + u = t$ and $\mathrm{ord}(s) + \mathrm{ord}(u) = \mathrm{ord}(t)$ (cf. [11]). The number of representations and of maximal representations of elements in a semigroup is related to some of the objects of our study; see [9] for more on factorization in numerical semigroups.

The book [25] is an exhaustive source on the subject of numerical semigroups.

We now give the main definitions of the paper.

Definitions 1.1. Let S be a numerical semigroup minimally generated by $g_1 < \cdots < g_\nu$. For each $i = 2, \ldots, \nu$ define:

$$\begin{aligned}
\tau_i &= \tau_i(S) = \min\{h \in \mathbb{N} \mid hg_i \in \langle g_1, \ldots, g_{i-1}\rangle\} - 1;\\
\alpha_i &= \alpha_i(S) = \max\{h \in \mathbb{N} \mid hg_i \in \mathrm{Ap}(S)\};\\
\beta_i &= \beta_i(S) = \max\{h \in \mathbb{N} \mid hg_i \in \mathrm{Ap}(S) \text{ and } \mathrm{ord}(hg_i) = h\};\\
\gamma_i &= \gamma_i(S) = \max\{h \in \mathbb{N} \mid hg_i \in \mathrm{Ap}(S) \text{ and } hg_i \text{ is the only maximal representation}\}.
\end{aligned}$$

If $\mathbf{n} = \{n_1, \ldots, n_\nu\}$ is any rearrangement of the minimal generators (i.e., the minimal system of generators not necessarily in increasing order), define for each $i = 2, \ldots, \nu$:

$$\phi_i = \phi_i(S, \mathbf{n}) = \min\{h \in \mathbb{N} \mid hn_i \in \langle n_1, \ldots, n_{i-1}\rangle\} - 1.$$

Remark 1.2. For each index $i = 2, \ldots, \nu$, we clearly have $\gamma_i \le \beta_i \le \alpha_i$. This, together with the fact that $\mathrm{Ap}(S) \subseteq \Big\{ \sum_{i=2}^{\nu} \lambda_i g_i \mid 0 \le \lambda_i \le \gamma_i \Big\}$ (cf. [14, Corollary 2.7]), implies that

$$\mathrm{Ap}(S) \subseteq \Big\{ \sum_{i=2}^{\nu} \lambda_i g_i \mid 0 \le \lambda_i \le \gamma_i \Big\} \subseteq \Big\{ \sum_{i=2}^{\nu} \lambda_i g_i \mid 0 \le \lambda_i \le \beta_i \Big\} \subseteq \Big\{ \sum_{i=2}^{\nu} \lambda_i g_i \mid 0 \le \lambda_i \le \alpha_i \Big\}.$$

In particular, we have $m = |\mathrm{Ap}(S)| \le \prod_{i=2}^{\nu} (\gamma_i + 1) \le \prod_{i=2}^{\nu} (\beta_i + 1) \le \prod_{i=2}^{\nu} (\alpha_i + 1)$.

Definitions 1.3. Let S be a numerical semigroup minimally generated by $g_1 < \cdots < g_\nu$.

(1) S is **telescopic** if $\mathrm{Ap}(S) = \Big\{ \sum_{i=2}^{\nu} \lambda_i g_i \mid 0 \le \lambda_i \le \tau_i \Big\}$;
(2) S is **associated to a plane branch** if S is telescopic and $(\tau_i + 1) g_i < g_{i+1}$ for all $i = 2, \ldots, \nu - 1$;

(3) S has **α-rectangular Apéry set** if $\mathrm{Ap}(S) = \left\{ \sum_{i=2}^{\nu} \lambda_i g_i \,|\, 0 \le \lambda_i \le \alpha_i \right\}$;
(4) S has **β-rectangular Apéry set** if $\mathrm{Ap}(S) = \left\{ \sum_{i=2}^{\nu} \lambda_i g_i \,|\, 0 \le \lambda_i \le \beta_i \right\}$;
(5) S has **γ-rectangular Apéry set** if $\mathrm{Ap}(S) = \left\{ \sum_{i=2}^{\nu} \lambda_i g_i \,|\, 0 \le \lambda_i \le \gamma_i \right\}$;
(6) S is **free** if there exists a rearrangement $\mathbf{n} = \{n_1, \dots, n_\nu\}$ of the minimal generators such that $\mathrm{Ap}(S, n_1) = \left\{ \sum_{i=2}^{\nu} \lambda_i n_i \,|\, 0 \le \lambda_i \le \phi_i \right\}$.

Notice that the definitions of telescopic and free semigroups are not standard, but it is proved in [25] that the conditions we state are equivalent to the classical definitions.

We turn now to the study of semigroups with α-rectangular Apéry set providing some characterizations, then we collect analogous statements for classes (1), (4), (5) and (6). In [23] Rosales introduced the following definition: a numerical semigroup S has **Apéry set of unique expression** if every element in $\mathrm{Ap}(S)$ has a unique representation. We will see that this condition is closely related to having α-rectangular Apéry set.

Lemma 1.4. *If $s \preceq t$ and $t \in \mathrm{Ap}(S)$, then $s \in \mathrm{Ap}(S)$.*

Proof. It follows from the definition of Apéry set. ☐

Lemma 1.5. *If $s \preceq t$ and t has a unique representation, then $s \preceq_M t$ and s has a unique representation.*

Proof. If an element has a unique representation then this must be maximal and the sum of the coefficients equals the order of the element. Let $t = \sum_{i=1}^{\nu} \lambda_i g_i$ and $s + u = t$ for some $u \in S$. Since the representation of t is unique, it follows that $s = \sum_{i=1}^{\nu} \xi_i g_i$ and $u = \sum_{i=1}^{\nu} \rho_i g_i$, with $\rho_i + \xi_i = \lambda_i$ for each i. These representations must be unique, otherwise t has a double representation, and we get $\mathrm{ord}(s) + \mathrm{ord}(u) = \sum_{i=1}^{\nu} \xi_i + \sum_{i=1}^{\nu} \rho_i = \sum_{i=1}^{\nu} \lambda_i = \mathrm{ord}(t)$. ☐

Proposition 1.6. *The following conditions are equivalent:*

(i) $\mathrm{Ap}(S)$ *is α-rectangular;*
(ii) there is only one maximal element in $(\mathrm{Ap}(S), \preceq)$ *and it has a unique representation;*
(iii) S is symmetric and $\mathrm{Ap}(S)$ *is of unique expression;*
(iv) $f + m = \sum_{i=2}^{\nu} \alpha_i g_i$;
(v) $m = \prod_{i=2}^{\nu} (\alpha_i + 1)$.

Proof. $(i) \Rightarrow (ii)$ Since $\mathrm{Ap}(S)$ is α-rectangular, we immediately get that $\sum_{i=2}^{\nu} \alpha_i g_i$ is the unique maximal element in $(\mathrm{Ap}(S), \preceq)$. Let us suppose that $\sum_{i=2}^{\nu} \alpha_i g_i = \sum_{i=2}^{\nu} u_i g_i$ for some non-negative integers u_i. By Lemma 1.4, $u_i g_i \in \mathrm{Ap}(S)$ for each i and hence $u_i \le \alpha_i$, by definition of α_i; it follows that $u_i = \alpha_i$ for each index i and the two representations coincide.

$(ii) \Leftrightarrow (iii)$ It follows by Lemma 1.5 and by the fact that S is symmetric if and only if $f + m$ is the only maximal element of $(\mathrm{Ap}(S), \preceq)$.

$(ii) \Rightarrow (iv)$ The unique maximal element in $(\mathrm{Ap}(S), \preceq)$ is necessarily $f + m$. Therefore $\alpha_i g_i \preceq f + m$ for each $i = 2, \dots, \nu$. Since $f + m$ has a unique representation, the thesis follows immediately.

$(iv) \Rightarrow (i)$ Since $f + m \in \mathrm{Ap}(S)$ in general, it follows by Lemma 1.4.

$(i) \Rightarrow (v)$ It follows by $m = |\mathrm{Ap}(S)|$ and by the fact that $\mathrm{Ap}(S)$ is of unique expression.

$(v) \Rightarrow (i)$ We already noticed that $\mathrm{Ap}(S) \subseteq \left\{ \sum_{i=2}^{\nu} \lambda_i g_i \,|\, 0 \le \lambda_i \le \alpha_i \right\}$ and since $m = \prod_{i=2}^{\nu}(\alpha_i+1) = |\mathrm{Ap}(S)|$, we must have an equality. □

Example 1.7. We apply the criterion above to show that the Apéry set of $S = \langle 12, 15, 16, 18\rangle$ is α-rectangular, without even computing the whole $\mathrm{Ap}(S)$. We determine the α_i's:

$$\begin{aligned} 2\cdot 15 &= 12+18 \in 12+S \\ 2\cdot 16 &= 32 \notin 12+S \\ 3\cdot 16 &= 4\cdot 12 \in 12+S \\ 2\cdot 18 &= 3\cdot 12 \in 12+S \end{aligned}$$

and so $\alpha_2 = 1$, $\alpha_3 = 2$, $\alpha_4 = 1$ and $m = 12 = 2\cdot 2\cdot 3 = \prod_{i=2}^{\nu}(\alpha_i+1)$.

Now we give the criteria for the remaining classes. A semigroup is called **M-pure** if all the maximal elements in the poset $(\mathrm{Ap}(S), \preceq_M)$ have the same order; M-pure semigroups were introduced in [11] along the way to the characterization of Gorenstein associated graded rings. In analogy to [23], we say that a semigroup S has **Apéry set of unique maximal expression** if every element in $\mathrm{Ap}(S)$ has a unique maximal representation. In connection to this, the number of maximal representations of elements in a semigroup has been investigated recently (cf. [12], [13]).

Proposition 1.8 ([14], Theorem 2.16)**.** *The following conditions are equivalent:*

(i) $\mathrm{Ap}(S)$ *is β-rectangular;*
(ii) S *is M-pure, symmetric and* $\mathrm{Ap}(S)$ *is of unique maximal expression;*
(iii) $\mathrm{Ap}(S)$ *has a unique maximal element with respect to* $\preceq_M$ *and this element has a unique maximal representation;*
(iv) $f + m = \sum_{i=2}^{\nu} \beta_i g_i$;
(v) $m = \prod_{i=2}^{\nu}(\beta_i+1)$.

Proposition 1.9 ([14], Theorem 2.22)**.** *The following conditions are equivalent:*

(i) $\mathrm{Ap}(S)$ *is γ-rectangular;*
(ii) $f + m = \sum_{i=2}^{\nu} \gamma_i g_i$;
(iii) $m = \prod_{i=2}^{\nu}(\gamma_i+1)$.

Proposition 1.10 ([25], Proposition 9.15)**.** *The following conditions are equivalent:*

(i) S *is telescopic;*
(ii) $f + m = \sum_{i=2}^{\nu} \tau_i g_i$;
(iii) $m = \prod_{i=2}^{\nu}(\tau_i+1)$.

Proposition 1.11 ([25], Proposition 9.15)**.** *The following conditions are equivalent:*

(i) S *is free;*
(ii) there is an arrangement $\mathbf{n}$ *of the minimal generators such that* $f + n_1 = \sum_{i=2}^{\nu} \phi_i n_i$;
(iii) there is an arrangement $\mathbf{n}$ *of the minimal generators such that* $n_1 = \prod_{i=2}^{\nu}(\phi_i+1)$.

The next lemma is a crucial step in establishing one of the implications in Theorem 1.13.

Lemma 1.12. *Let S have γ-rectangular Apéry set. For each $i = 2, \dots, \nu$ there exist relations*

$$(\star) \qquad (\gamma_i+1)g_i = \lambda_{i,1}g_1 + \lambda_{i,2}g_2 + \cdots + \lambda_{i,\nu}g_\nu$$

and a permutation σ *of* $\{1, \dots, \nu\}$ *such that* $\sigma(1) = 1$ *and* $\lambda_{\sigma(i),\sigma(j)} = 0$ *if* $i \le j$, $j \ge 2$.

Proof. Fix an index $i \in \{2, \dots, \nu\}$. By definition of γ_i we have two possible cases:

(I) $(\gamma_i + 1)g_i \in \mathrm{Ap}(S)$. Then the representation $(\gamma_i + 1)g_i$ is not maximal or it is not the unique maximal one; hence there is a different representation $\sum_{j=1}^{\nu} \lambda_{i,j} g_j$ of the same element with $\gamma_i + 1 \le \sum_{j=1}^{\nu} \lambda_{i,j}$. Notice that $(\gamma_i + 1)g_i \in \mathrm{Ap}(S)$ forces $\lambda_{i,1} = 0$.

(II) $(\gamma_i + 1)g_i \notin \mathrm{Ap}(S)$. Then we can write $(\gamma_i + 1)g_i = \sum_{j=1}^{\nu} \lambda_{i,j} g_j$ for some non-negative integers $\lambda_{i,j}$, with $\lambda_{i,1} > 0$.

It is useful to consider the square matrix obtained from the relations ($\star$) found in (I) and (II) leaving out the coefficients of g_1

$$\mathbf{L} = \begin{pmatrix} \lambda_{2,2} & \lambda_{2,3} & \dots & \lambda_{2,\nu} \\ \lambda_{3,2} & \lambda_{3,3} & \dots & \lambda_{3,\nu} \\ \dots & \dots & \dots & \dots \\ \lambda_{\nu,2} & \lambda_{\nu,3} & \dots & \lambda_{\nu,\nu} \end{pmatrix}.$$

Now we construct a permutation σ of $\{1, 2, \dots, \nu\}$ satisfying $\sigma(1) = 1$ and $\lambda_{\sigma(i),\sigma(j)} = 0$ whenever $i \le j$ and $j \ge 2$, or equivalently such that the square matrix

$$\mathbf{L}_\sigma = \begin{pmatrix} \lambda_{\sigma(2),\sigma(2)} & \lambda_{\sigma(2),\sigma(3)} & \dots & \lambda_{\sigma(2),\sigma(\nu)} \\ \lambda_{\sigma(3),\sigma(2)} & \lambda_{\sigma(3),\sigma(3)} & \dots & \lambda_{\sigma(3),\sigma(\nu)} \\ \dots & \dots & \dots & \dots \\ \lambda_{\sigma(\nu),\sigma(2)} & \lambda_{\sigma(\nu),\sigma(3)} & \dots & \lambda_{\sigma(\nu),\sigma(\nu)} \end{pmatrix}$$

is lower triangular with zeros in the diagonal. We proceed by decreasing induction on h.

For the basis of the induction $h = \nu$ it is enough to show that there exists a column in $\mathbf{L}$ with all zero entries. Let us suppose by contradiction that every column in $\mathbf{L}$ has a non zero element, that is, for every $j \ge 2$ there exists $\tau(j)$ such that $\lambda_{\tau(j),j} > 0$. Taking the sum over all the relations ($\star$) we obtain

$$\sum_{i=2}^{\nu} (\gamma_i + 1) g_i = \sum_{i=2}^{\nu} \sum_{j=1}^{\nu} \lambda_{i,j} g_j.$$

and subtracting $\sum_{i=2}^{\nu} g_i$ from both sides we get

$$u := \sum_{i=2}^{\nu} \gamma_i g_i = \sum_{j=1}^{\nu} \sum_{i \neq 1, \tau(j)}^{\nu} \lambda_{i,j} g_j + \sum_{i=2}^{\nu} (\lambda_{\tau(i),i} - 1) g_i.$$

As $u \in \mathrm{Ap}(S)$ by γ-rectangularity, we necessarily have $\lambda_{i,1} = 0$ and hence case (II) above is not possible for any $i \in \{2, \dots, \nu\}$. We get by (I) that $\sum_{j=1}^{\nu} \lambda_{i,j} \ge \gamma_i + 1$ for every i. Furthermore, the representation $u = \sum_{i=2}^{\nu} \gamma_i g_i$ is maximal by [14, Lemma 2.19] and so if there exists i such that $\sum_{j=1}^{\nu} \lambda_{i,j} > \gamma_i + 1$ then it follows

$$\mathrm{ord}(u) = \sum_{i=2}^{\nu} \gamma_i < \sum_{j=1}^{\nu} \sum_{i \neq 1, \tau(j)}^{\nu} \lambda_{i,j} + \sum_{i=2}^{\nu} (\lambda_{\tau(j),j} - 1) \le \mathrm{ord}(u)$$

yielding a contradiction; thus $\sum_{j=1}^{\nu} \lambda_{i,j} = \gamma_i + 1$ for every i. In particular for the index of the largest generator we have

$$(\gamma_\nu + 1)g_\nu = \sum_{j=1}^{\nu} \lambda_{\nu,j} g_j \quad \text{and} \quad \sum_{j=1}^{\nu} \lambda_{\nu,j} = \gamma_\nu + 1.$$

But $g_j < g_\nu$ for $j \neq \nu$ forces $\lambda_{\nu,j} = 0$ and $\lambda_{\nu,\nu} = \gamma_\nu + 1$, contradicting the fact that in (I) we found a different representation. So the p-th column of $\mathbf{L}$ consists of zeros for some $p \geq 2$, and we let $\sigma(\nu) = p$.

Now let $1 < h < \nu$ and suppose that for every $j \in \{\sigma(\nu), \sigma(\nu-1), \ldots, \sigma(h+1)\}$ and $i \leq j$ we have $\lambda_{\sigma(i),\sigma(j)} = 0$. By repeating the same argument as in the basis of the induction for the submatrix of $\mathbf{L}$ indexed by $i, j \in \{2, \ldots, \nu\} \setminus \{\sigma(\nu), \sigma(\nu-1), \ldots, \sigma(h+1)\}$ we get a new index $\sigma(h)$ for which the statement is true, and the inductive step follows. $\square$

In order to present the main theorem of the paper, we need to give one more definition. A numerical semigroup S has a **unique Betti element** if the first syzygies of the semigroup ring $\Bbbk[[t^S]]$ have all the same degree (in the S-grading; see [18] for a purely numerical definition). In [18] the authors prove that $S = \langle g_1, \ldots, g_\nu \rangle$ has a unique Betti element if and only if there exist pairwise coprime integers $a_1, \ldots, a_\nu$ greater than one such that $g_i = \prod_{j \neq i} a_i$; these semigroups are shown to be complete intersection. Moreover in [5] it is shown that for such a semigroup S the tangent cone of the semigroup ring $\Bbbk[[t^S]]$ is a complete intersection, implying thus that $\mathrm{Ap}(S)$ is γ-rectangular by [14, Theorem 3.6].

Theorem 1.13. *Let S be a numerical semigroup. Consider the following conditions:*

(1) S is associated to a plane branch;
(2) S has a unique Betti element;
(3) S is telescopic;
(4) S has α-rectangular Apéry set;
(5) S has β-rectangular Apéry set;
(6) S has γ-rectangular Apéry set;
(7) S is free;
(8) S is complete intersection.

Then (1) $\Rightarrow$ (3) $\Rightarrow$ (5), (2) $\Rightarrow$ (4) $\Rightarrow$ (5), (1) $\Rightarrow$ (4), (2) $\Rightarrow$ (3), (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) *(compare Figure 1). Moreover, all the implications are strict.*

Proof. In each of the proofs below, let S be minimally generated by $g_1 < \cdots < g_\nu$.
• Plane branch $\Rightarrow$ Telescopic.
It follows from Definitions 1.3. The semigroup $S = \langle 6, 10, 15 \rangle$ is not associated to a plane branch, as $(\tau_2 + 1)g_2 = 3 \cdot 10 > 15 = g_3$; however S has a unique Betti element, in particular it is telescopic and with α-rectangular Apéry set (see below).
• Plane branch $\Rightarrow$ α-rectangular Apéry set.
We prove that $(\tau_i + 1)g_i \notin \mathrm{Ap}(S)$ by induction on $i \in \{2, \ldots, \nu\}$. Since $(\tau_2 + 1)g_2 \in \langle g_1 \rangle$ we get $(\tau_2 + 1)g_2 \notin \mathrm{Ap}(S)$. Given $i > 2$, we have $(\tau_i + 1)g_i = \lambda_1 g_1 + \cdots + \lambda_{i-1} g_{i-1}$ for some $\lambda_j \in \mathbb{N}$. Assume by contradiction $(\tau_i + 1)g_i \in \mathrm{Ap}(S)$, then by induction and Lemma 1.4 we must have $\lambda_1 = 0$ and $\lambda_j \leq \tau_j$ for $j = 2, \ldots, i-1$. By definition of semigroup associated to

a plane branch, we have the following chain of inequalities:

$$\begin{aligned}(\tau_i+1)g_i &\geq 2g_i > 2(\tau_{i-1}+1)g_{i-1} \geq (\tau_{i-1}+1)g_{i-1}+2g_{i-1} > \\ &> (\tau_{i-1}+1)g_{i-1}+2(\tau_{i-2}+1)g_{i-2} \geq \cdots \geq \\ &\geq (\tau_{i-1}+1)g_{i-1}+\cdots+(\tau_2+1)g_2 > \lambda_1 g_1+\cdots+\lambda_{i-1}g_{i-1} = (\tau_i+1)g_i\end{aligned}$$

reaching a contradiction. Hence $(\tau_i+1)g_i \notin \mathrm{Ap}(S)$ and $\alpha_i \leq \tau_i$. Finally $\mathrm{Ap}(S)$ is α-rectangular by Proposition 1.6 (v) as $m \leq \prod_{i=2}^{\nu}(\alpha_i+1) \leq \prod_{i=2}^{\nu}(\tau_i+1) = m$, where we used Remark 1.2 and the fact that S is telescopic.
• Unique Betti element $\Rightarrow$ α-rectangular Apéry set.
Let $a_1 > a_2 > \cdots > a_\nu > 1$ be pairwise coprime integers such that $g_i = \prod_{j\neq i} a_j$. Similarly to the previous proof, it suffices to show that $\alpha_i + 1 \leq a_i$. But this is trivial as $a_i g_i = a_1 g_1 \notin \mathrm{Ap}(S)$. Now let $S = \langle 4, 6, 13\rangle$: we have $\mathrm{Ap}(S) = \{0, 6, 13, 19\}$, $\tau_2 = \tau_3 = 1$, and $m = (\tau_2+1)(\tau_3+1)$, $(\tau_2+1)g_2 < g_3$. So S is associated to a plane branch and hence telescopic and with α-rectangular Apéry set, but S does not have a unique Betti element.
• Unique Betti element $\Rightarrow$ Telescopic.
Let $a_1 > a_2 > \cdots > a_\nu > 1$ be pairwise coprime integers such that $g_i = \prod_{j\neq i} a_j$. We show that $\tau_i = a_i - 1$ for each $i \geq 2$, from which it follows that S is telescopic by Proposition 1.10 (iii). Since the a_j's are coprime, a_i does not divide hg_i for $h \leq a_i - 1$, hence $hg_i \notin \langle g_1, \ldots, g_{i-1}\rangle$. However $a_i g_i = a_1 g_1 \in \langle g_1, \ldots, g_{i-1}\rangle$ so that $\tau_i = a_i - 1$.
• α-rectangular Apéry set $\Rightarrow$ β-rectangular Apéry set $\Rightarrow$ γ-rectangular Apéry set.
It follows from Remark 1.2. The semigroup $S = \langle 8, 10, 15\rangle$ is telescopic and therefore $\mathrm{Ap}(S)$ is β-rectangular (see below), but it is not α-rectangular: $\mathrm{Ap}(S) = \{0, 10, 15, 20, 25, 30, 35, 45\}$ and it is easy to check that $\alpha_2 = \alpha_3 = 3$ and $\tau_2 = 3, \tau_3 = 1$ so that $m = (\tau_2+1)(\tau_3+1)$ but $m \neq (\alpha_2+1)(\alpha_3+1)$. The Apéry set of $S = \langle 8, 10, 11, 12\rangle$ is γ-rectangular but not β-rectangular: we have $\mathrm{Ap}(S) = \{0, 10, 11, 12, 21, 22, 23, 33\}$ and we get $\beta_2 = 1, \beta_3 = 3, \beta_4 = 1, \gamma_2 = \gamma_3 = \gamma_4 = 1$, hence $m = (\gamma_2+1)(\gamma_3+1)(\gamma_4+1)$ and $m \neq (\beta_2+1)(\beta_3+1)(\beta_4+1)$.
• Telescopic $\Rightarrow$ β-rectangular Apéry set.
For each $i \in \{2, \ldots, \nu\}$ we have $(\tau_i+1)g_i = \lambda_1 g_1 + \cdots + \lambda_{i-1}g_{i-1}$ for some $\lambda_j \in \mathbb{N}$. The fact that $g_1 < \cdots < g_{i-1} < g_i$ forces $\lambda_1 + \cdots + \lambda_{i-1} > \tau_i + 1$ and therefore $\mathrm{ord}((\tau_i+1)g_i) > \tau_i + 1$. It follows that $\beta_i \leq \tau_i$ and by Remark 1.2 we get $m \leq \prod_{i=2}^{\nu}(\beta_i+1) \leq \prod_{i=2}^{\nu}(\tau_i+1) = m$ and hence $\mathrm{Ap}(S)$ is β-rectangular by Proposition 1.8 (v). Let $S = \langle 4, 5, 6\rangle$: we have $\mathrm{Ap}(S) = \{0, 5, 6, 11\}$ and thus $\alpha_2 = \alpha_3 = 1$, $\tau_2 = 3, \tau_3 = 1$ so that $\mathrm{Ap}(S)$ is α-rectangular as $m = (\alpha_2+1)(\alpha_3+1)$ (hence β-rectangular) but S is not telescopic as $m \neq (\tau_2+1)(\tau_3+1)$.
• γ-rectangular Apéry set $\Rightarrow$ Free.
Assume S has γ-rectangular Apéry set. Let σ be the permutation of $\{1, \ldots, \nu\}$ as in Lemma 1.12, and consider the rearrangement of the minimal generators $\mathbf{n} = \{n_1, \ldots, n_\nu\}$ with $n_i = g_{\sigma(i)}$. By relations $(\star)$ for each $i = 2, \ldots, \nu$ we get

$$(\gamma_{\sigma(i)}+1)n_i = (\gamma_{\sigma(i)}+1)g_{\sigma(i)} = \sum_{j=1}^{\nu}\lambda_{\sigma(i),j}g_j = \sum_{j=1}^{\nu}\lambda_{\sigma(i),\sigma(j)}g_{\sigma(j)} = \sum_{j=1}^{\nu}\lambda_{\sigma(i),\sigma(j)}n_j$$

thus $\phi_i \leq \gamma_{\sigma(i)}$ by the triangularity of the matrix $\mathbf{L}_\sigma$. Following the notation of [25], let

$$\overline{c}_i = \min\left\{h \in \mathbb{N}\setminus\{0\} \;\middle|\; \gcd(n_1, \ldots, n_{i-1}) \text{ divides } hn_i\right\}.$$

In [25, Lemma 9.13] it is proved that $n_1 = \prod_{i=2}^{\nu} \overline{c}_i$ and $\overline{c}_i \leq \phi_i + 1$. On the other hand $n_1 = \prod_{i=2}^{\nu}(\gamma_i + 1) = \prod_{i=2}^{\nu}(\gamma_{\sigma(i)} + 1)$ by Proposition 1.9 (*iii*). We conclude that

$$n_1 = \prod_{i=2}^{\nu} \overline{c}_i \leq \prod_{i=2}^{\nu}(\phi_i + 1) \leq \prod_{i=2}^{\nu}(\gamma_{\sigma(i)} + 1) = n_1$$

hence $n_1 = \prod_{i=2}^{\nu}(\phi_i + 1)$ and S is free by Proposition 1.11 (*iii*).

Let $S = \langle 5, 6, 9\rangle$. Since 5 is prime, we cannot have $m = (\gamma_2 + 1)(\gamma_3 + 1)$, therefore $\mathrm{Ap}(S)$ is not γ-rectangular. Consider the arrangement $\mathbf{n} = \{6, 9, 5\}$: we have $\phi_2 = 1, \phi_3 = 2$ so that S is free as $n_1 = (\phi_2 + 1)(\phi_3 + 1)$.

• Free $\Rightarrow$ Complete intersection.

This is well-known and is proven e.g. in [25, Corollary 9.17] by means of gluing. Counterexamples for the inverse implication are provided at the beginning of the next section. $\square$

2. Gluing and other applications

In this section we explore an operation that allows to construct new (more complicated) semigroups from old ones. Let S_1 and S_2 be two numerical semigroups minimally generated by $n_1, \ldots, n_r$ and $m_1, \ldots, m_s$, respectively. Given positive integers $d_1 \in S_1 \setminus \{n_1, \ldots, n_r\}$ and $d_2 \in S_2 \setminus \{m_1, \ldots, m_s\}$ such that $\gcd(d_1, d_2) = 1$, the semigroup

$$S = d_2 S_1 + d_1 S_2 = \langle d_2 n_1, \ldots, d_2 n_r, d_1 m_1, \ldots, d_1 m_s\rangle$$

is called a **gluing** of S_1 and S_2. Notice that $\nu(S) = \nu(S_1) + \nu(S_2)$. The importance of gluing was first highlighted in [16], where the author proved that a semigroup is a complete intersection if and only if it is a gluing of two complete intersection semigroups, formulating thus a recursive characterization. An analogous statement holds for symmetric semigroups. Although the gluing of two free semigroups needs not be free, a semigroup of embedding dimension ν is free if and only if it is a gluing of $\mathbb{N}$ and a free semigroup of embedding dimension $\nu - 1$ (cf. [25, Theorem 9.16]). We remark that gluing has other interesting applications, including partial positive answers to Rossi's conjecture (cf. [3], [20]).

Example 2.1. As an illustration, we construct a family of complete intersection semigroups that are not free. Let p_1, p_2, p_3, p_4 be distinct primes such that $p_3, p_4 > p_1 p_2$. Consider

$$S = \langle p_1 p_3, p_2 p_3, p_1 p_4, p_2 p_4\rangle = d_2 T + d_1 T$$

where $T = \langle p_1, p_2\rangle$, $d_1 = p_4$ and $d_2 = p_3$ (note $p_3, p_4 \in T \setminus \{p_1, p_2\}$ as $f(T) = p_1 p_2 - p_1 - p_2$). Now T is a complete intersection being two-generated, therefore S is a complete intersection. However, there is no hope of expressing S as a gluing of $\mathbb{N}$ and a three-generated semigroup because any three generators of S are coprime; by the characterization above S is not free.

Remark 2.2. By [25, Theorem 9.16] and by definition, it is easy to see that a semigroup S is telescopic if and only if it is a gluing of $\mathbb{N}$ and a telescopic semigroup $T = \langle n_1, \ldots, n_{\nu-1}\rangle$ with $d_2 > d_1 n_{\nu-1}$.

Furthermore, it is also easy to check that a semigroup $S = \langle g_1, \ldots, g_\nu\rangle$ has a unique Betti element if and only if it is the gluing $d_1 T + d_2 \mathbb{N}$ where $T = \langle n_1, \ldots, n_{\nu-1}\rangle$ has a unique Betti element, $d_2 = \mathrm{lcm}(n_1, \ldots, n_{\nu-1})$ and $\gcd(n_i, d_1) = 1$ for each i.

Finally, by definition, a semigroup is associated to a plane branch if and only if it is a gluing of $\mathbb{N}$ and a semigroup associated to a plane branch $T = \langle n_1, \dots, n_{\nu-1}\rangle$ with $d_2 > d_1(\tau_{\nu-1}(T)+1)n_{\nu-1}$.

Our aim at this point is to push this study further: we use gluing to prove a recursive characterization for semigroups with α-rectangular Apéry sets.

Theorem 2.3. *Let T be a semigroup with α-rectangular Apéry set and $d_1, d_2 \in \mathbb{N}$ such that $d_1 \notin \mathrm{Ap}(T)$, $d_1 > d_2 m(T)$; then the gluing $S = d_2T + d_1\mathbb{N}$ has α-rectangular Apéry set. Conversely, every semigroup $S \neq \mathbb{N}$ with α-rectangular Apéry set arises in this way.*

Proof. Assume that S is the gluing $d_2T + d_1\mathbb{N}$ where $T = \langle n_1 < \cdots < n_{\nu-1}\rangle$ has α-rectangular Apéry set and $d_1 \in T\setminus\{n_1, \dots, n_{\nu-1}\}$, $d_2 \in \mathbb{N}\setminus\{1\}$ are coprime integers satisfying $d_1 \notin \mathrm{Ap}(T)$ and $d_1 > d_2m(T)$; in particular we have $m(S) = d_2m(T)$. In the proof of this implication $\alpha_i(S)$ denotes, with an abuse of notation, the integer α from Defintions 1.1 relative to the minimal generator d_2n_i of S (which is not necessarily the i-th generator of S in increasing order). By Proposition 1.6 (*v*), $n_1 = \prod_{i=2}^{\nu-1}(\alpha_i(T)+1)$. We claim that $\alpha_i(S) \leq \alpha_i(T)$ for each $i = 2, \dots, \nu-1$. In fact

$$(\alpha_i(T)+1)n_i = \lambda_1 n_1 + \cdots + \lambda_{\nu-1}n_{\nu-1} \implies (\alpha_i(T)+1)d_2n_i = \lambda_1 d_2 n_1 + \cdots + \lambda_{\nu-1}d_2n_{\nu-1}$$

for some $\lambda_j \in \mathbb{N}$ with $\lambda_1 > 0$. Since $m(S) = d_2n_1$ we get $(\alpha_i(T)+1)d_2n_i \notin \mathrm{Ap}(S)$, proving that $\alpha_i(S) \leq \alpha_i(T)$. Now we show that $\alpha_\nu(S) \leq d_2 - 1$: we have $d_1 - n_1 \in T$ as $d_1 \notin \mathrm{Ap}(T)$, therefore $d_2d_1 - d_2n_1 \in S$ and $d_2d_1 \notin \mathrm{Ap}(S)$. By Remark 1.2

$$m(S) \leq \prod_{i=2}^{\nu}(\alpha_i(S)+1) \leq d_2\prod_{i=2}^{\nu-1}(\alpha_i(T)+1) = d_2n_1 = m(S)$$

and hence $\mathrm{Ap}(S)$ is α-rectangular by $m(S) = \prod_{i=2}^{\nu}(\alpha_i(S)+1)$.

Assume now that $S = \langle g_1 < \cdots < g_\nu\rangle \neq \mathbb{N}$ has α-rectangular Apéry set. By Theorem 1.13 $\mathrm{Ap}(S)$ is γ-rectangular and thus there is a rearrangement $\mathbf{n} = \{n_1, \dots, n_\nu\}$ of the minimal generators such that $g_1 = n_1$ and fulfilling the conditions of Proposition 1.11; let σ be the permutation such that $n_i = g_{\sigma(i)}$. Let $d = \gcd(n_1, \dots, n_{\nu-1})$. Then S is the gluing of $T = \left\langle \frac{n_1}{d}, \dots, \frac{n_{\nu-1}}{d}\right\rangle$ and $\mathbb{N}$, with integers $d_1 = n_\nu$ and $d_2 = d$; furthermore T is free by [25, Theorem 9.16]. We prove that $\mathrm{Ap}(T)$ is α-rectangular.

Let $l = \sigma(\nu)$; it is shown in [25, Lemma 9.13 (3), Proposition 9.15 (4)] that

$$d = \min\left\{h \in \mathbb{N} \,\middle|\, hg_l \in \langle g_1, \dots, \widehat{g_l}, \dots, g_\nu\rangle\right\}.$$

By unique expression of $\mathrm{Ap}(S)$ we get $hg_l \notin \langle g_1, \dots, \widehat{g_l}, \dots, g_\nu\rangle$ for all $h \leq \alpha_l(S)$, so $\alpha_l(S) \leq d-1$. On the other hand $(\alpha_l(S)+1)g_l \notin \mathrm{Ap}(S)$, so it has another representation involving the multiplicity g_1, and by maximality of $\alpha_l(S)$ this representation does not involve g_l. Thus $(\alpha_l(S)+1)g_l \in \langle g_1, \dots, \widehat{g_l}, \dots, g_\nu\rangle$ and $\alpha_l(S)+1 \geq d$. Hence $d = \alpha_l(S)+1$.

Let us show now that $\alpha_i(T) \leq \alpha_{\sigma(i)}(S)$ for each $i = 2, \dots, \nu$ (here $\alpha_i(T)$ denotes the integer α relative to the minimal generator $\frac{n_i}{d}$ of T). If $\alpha_i(T) > \alpha_{\sigma(i)}(S)$, then

$$(\alpha_{\sigma(i)}(S)+1)\frac{g_{\sigma(i)}}{d} \in \mathrm{Ap}(T) \implies (\alpha_{\sigma(i)}(S)+1)\frac{g_{\sigma(i)}}{d} - \frac{g_1}{d} \notin T$$

because $m(T) = \frac{g_1}{d}$. By definition of $\alpha_{\sigma(i)}$, we have

$$(\alpha_{\sigma(i)}(S)+1)g_{\sigma(i)} - g_1 \in S \implies (\alpha_{\sigma(i)}(S)+1)g_{\sigma(i)} = g_1 + \sum_{j\neq l} \xi_j g_j + \xi_l g_l$$

hence d divides $\xi_l g_l$, but $\gcd(d, g_l) = \gcd(S) = 1$, therefore d actually divides ξ_l. It follows

$$(\alpha_{\sigma(i)}(S)+1)\frac{g_{\sigma(i)}}{d} = \frac{g_1}{d} + \sum_{j\neq l} \xi_j \frac{g_j}{d} + \frac{\xi_l}{d} g_l. \tag{†}$$

By definition of gluing $g_l = d_1 \in T \setminus \left\{\frac{n_1}{d}, \ldots, \frac{n_{\nu-1}}{d}\right\}$, i.e. $g_l = \sum_{j\neq l} \eta_j \frac{g_j}{d}$. Substituting this last equation in (†) we obtain the contradiction

$$(\alpha_{\sigma(i)}(S)+1)\frac{g_{\sigma(i)}}{d} - \frac{g_1}{d} = \sum_{j\neq l} \left(\xi_j + \frac{\eta_j \xi_l}{d}\right) \frac{g_j}{d} \in T.$$

Putting all the inequalities together, we get by Remark 1.2 and α-rectangularity of $\mathrm{Ap}(S)$

$$m(T) \leq \prod_{i=2}^{\nu-1} \Big(\alpha_i(T)+1\Big) \leq \prod_{i\neq l} \Big(\alpha_{\sigma(i)}(S)+1\Big) = \frac{\prod_{i=2}^{\nu} \Big(\alpha_i(S)+1\Big)}{\alpha_l(S)+1} = \frac{m(S)}{d} = m(T)$$

concluding that $\mathrm{Ap}(T)$ is α-rectangular by Proposition 1.6 (v).

Now if $d_1 = n_\nu \in \mathrm{Ap}(T)$, then $n_\nu = \sum_{i=2}^{\nu-1} \lambda_i \frac{n_i}{d}$ with $\lambda_i \leq \alpha_i(T)$ and hence $n_\nu = \sum_{i=2}^{\nu-1} \lambda_i \frac{g_{\sigma(i)}}{d}$ with $\lambda_i \leq \alpha_i(T) \leq \alpha_{\sigma(i)}(S)$, by the previous part of the proof. Since $\mathrm{Ap}(S)$ is α-rectangular, it follows that $dn_\nu \in \mathrm{Ap}(S)$, contradicting, again by the previous part of the proof, the definition of α_l.

The fact that $d_1 = n_\nu > d_2 m(T)$ follows from $d_2 m(T) = n_1 = g_1 < g_l = n_\nu$. □

Example 2.4. Let $T = \langle 18, 21, 27, 35\rangle$, then it is possible to check that $\alpha_2 = 2, \alpha_3 = 1, \alpha_4 = 2$ from which it follows that $\mathrm{Ap}(T)$ is α-rectangular. Let $S = 2T + 69\mathbb{N} = \langle 36, 42, 54, 69, 70\rangle$; then we have $\alpha_2 = 2, \alpha_3 = 1, \alpha_4 = 3, \alpha_5 = 2$ so that $\mathrm{Ap}(S)$ is not α-rectangular. This example shows the property of having α-rectangular Apéry set is not preserved by gluing with $\mathbb{N}$ if we drop the hypothesis $d_1 \notin \mathrm{Ap}(T)$ in Theorem 2.3 (notice that $69 \in \mathrm{Ap}(T)$).

Question 2.5. *Is it possible to characterize semigroups with β-rectangular and γ-rectangular Apéry set in terms of gluing?*

We conclude the paper by relating our work to a theorem of Watanabe and one of Rosales and Branco. In [28, Theorem 1] the author proves that there exist complete intersection semigroups S with prescribed values of multiplicty and embedding dimension, satisfying the condition $m(S) \geq 2^{\nu(S)-1}$. We want to apply Theorem 2.3 to prove a similar statement for semigroups with α-rectangular Apéry set. However we need the stronger condition $\ell(m(S)) \geq \nu(S) - 1$, where $\ell(\cdot)$ denotes the length of the factorization into primes of an integer. Note that this condition is implied if $\mathrm{Ap}(S)$ is α-rectangular, as it follows from Propostion 1.6 (v).

Corollary 2.6. *Given $m, \nu \in \mathbb{N}$ with $\ell(m) \geq \nu - 1, \nu \geq 2$, there exists a semigroup S with $m(S) = m$, $\nu(S) = \nu$ such that $\mathrm{Ap}(S)$ is α-rectangular.*

Proof. Since $\ell(m) \geq \nu-1$ we can write $m = a_1 a_2 \cdots a_{\nu-1}$, with $a_i \geq 2$ integers, not necessarily prime. Let $S^{(1)} = \langle a_1, b\rangle$ where $b > a_1$ and $\gcd(a_1, b) = 1$, then $\mathrm{Ap}(S^{(1)})$ is α-rectangular. Assume we constructed $S^{(j-1)}$ with $j > 1$ fulfilling $m(S^{(j-1)}) = a_1 \cdots a_{j-1}$, $\nu(S^{(j-1)}) = j$ and with $\mathrm{Ap}(S^{(j-1)})$ α-rectangular. Glue $S^{(j-1)}$ and $\mathbb{N}$ with integers d_1 and $d_2 = a_j$, choosing d_1 sufficiently large. By Theorem 2.3 the result $S^{(j)}$ has still α-rectangular Apéry set, and furthermore $m(S^{(j)}) = a_1 \cdots a_j$ and $\nu(S^{(j)}) = j + 1$. Finally take $S = S^{(\nu-1)}$. $\square$

Now we analyze a family of semigroups introduced in [24], where the authors provide families of free semigroups with arbitrary embedding dimension.

Proposition 2.7. *Let $a, b, p \in \mathbb{N}$ be such that $\gcd(a, b) = 1$ and $a, b, p > 1$. The semigroup $S = \langle a^p, a^p + b, a^p + ab, \ldots, a^p + a^{p-1}b\rangle$ has α-rectangular Apéry set and is not telescopic.*

Proof. Denote the minimal generators by $g_1 = a^p, \ldots, g_{p+1} = a^p+a^{p-1}b$ and let $i \in \{2, \ldots, p+1\}$. If $i \leq p$, we have $ag_i - g_1 = a(a^p + a^{i-2}b) - a^p = (a-1)a^p + a^{i-1}b = g_{i+1} + (a-2)g_1 \in S$. If $i = p + 1$ then $ag_i - g_1 = a(a^p + a^{p-1}b) - a^p = (a + b - 1)g_1 \in S$. In both cases we have $ag_i \notin \mathrm{Ap}(S)$ and hence $\alpha_i \leq a - 1$. Thus $g_1 = a^p \geq \prod_{i=2}^{p+1}(\alpha_i + 1)$. But we have in general $g_1 = a^p \leq \prod_{i=2}^{p+1}(\alpha_i + 1)$ (cf. Remark 1.2) and therefore $\mathrm{Ap}(S)$ is α-rectangular by Proposition 1.6 (v). These semigroups are never telescopic: since $\gcd(a, b) = 1$ we necessarily have $\tau_2 + 1 = a^p$ and since $p > 1$ it follows that $\prod_{i=2}^{p+1}(\tau_i + 1) > \tau_2 + 1 = a^p = m$. $\square$

References

1. R. Apéry, *Sur les branches superlinéaires des courbes algébriques*, C. R. Acad. Sci. Paris **222** (1946), 1198–1200.
2. A. Assi, P. A. García Sanchez, *Constructing the set of complete intersection numerical semigroups with a given Frobenius number*, to appear in Applicable Algebra in Engineering, Communication and Computing (2013). `arXiv:1204.4258`
3. F. Arslan, P. Mete, M. Sahin, *Gluing and Hilbert functions of monomial curves*, Proceedings of the American Mathematical Society **137** No. 7 (2009), 2225–2232.
4. V. Barucci, M. D'Anna, R. Fröberg, *On plane algebroid curves*, Commutative Ring Theory and Applications, Dekker Lecture Notes in Pure and Applied Mathematics, **231**, (2003), 37–50.
5. V. Barucci, R. Fröberg, *Associated graded rings of one dimensional analytically irreducible rings*, Journal of Algebra **304** No. 1 (2006), 349–358.
6. J. Bertin, P. Carbonne, *Semi-groupes d'entiers et application aux branches*, J. Algebra **49** (1977), 81–95.
7. I. Bermejo, P. Giminez, E. Reyes, R. H. Villarreal, *Complete intersections in affine monomial curves*, Boletin Sociedad Matemática Mexicana **3** No. 11 (2005), 191–203.
8. I. Bermejo, I. Garcia-Marco, J. J. Salazar-Gonzales, *An algorithm for checking whether the toric ideal of an affine monomial curve is a complete intersection*, J. Symbolic Computation **42** (2007), 971–991.
9. C. Bowles, S. T. Chapman, N. Kaplan, D. Reiser, *On Delta Sets of Numerical Monoids*, Journal of Algebra and its Applications **5** (2006), 695–718.
10. H. Bresinsky, *Semigroups corresponding to algebroid branches in the plane*, Proceedings of the American Mathematical Society **32** No. 2 (1972), 381-384.
11. L. Bryant, *Goto numbers of a numerical semigroup ring and the Gorensteiness of associated graded rings*, Communications in Algebra **38** No. 6 (2010), 2092–2128.
12. L. Bryant, J. Hamblin, *The maximal denumerant of a numerical semigroup*, to appear in Semigroup Forum, (2012). `DOI: 10.1007/s00233-012-9448-5`
13. L. Bryant, J. Hamblin, L. Jones, *Maximal denumerant of a numerical semigroup with embedding dimension less than four*, to appear in Journal of Commutative Algebra, (2012).
14. M. D'Anna, V. Micale, A. Sammartano, *When the associated graded ring of a semigroup ring is complete intersection*, Journal of Pure and Applied Algebra **217** (2013), 1007–1017.

15. M. Delgado, P. A. Garcia-Sanchez, J. Morais, *NumericalSgps - a GAP package*, Version 0.971 (2011).
16. C. Delorme, *Sous-monoïdes d'intersection complète de N*, Ann. Sci. Ecole Norm. Sup. **4** (1976), 145–154.
17. The GAP Group, *GAP - Groups, Algorithms, and Programming*, Version 4.4.10 (2007).
18. P. A. Garcia-Sanchez, I. Ojeda, J. C. Rosales, *Affine semigroups having a unique Betti number*, Journal of Algebra and its Applications **12** No. 3 (2013).
19. J. Herzog, *Generators and relations of abelian semigroup rings*, Manuscripta Math. **3** (1970), 748-751.
20. R. Jafari, S. Zarzuela Armengou, *On monomial curves obtained by gluing*, preprint (2013). `arXiv:1303.3789`
21. C. Kirfel, R. Pellikaan, *The minimum distance of codes in an array coming from telescopic semigroups*, IEEE Transactions on information theory, **41** No. 6, (1995), 1720–1732.
22. V. Micale, A. G. Olteanu, *On the Betti numbers of some semigroup rings*, Le Matematiche **67** (2012), 145–159.
23. J. C. Rosales, *Numerical semigroups with Apéry sets of unique expression*, Journal of Algebra **226** No. 1 (2000), 479–487.
24. J. C. Rosales, M. B. Branco, *Three families of free numerical semigoups with arbitrary embedding dimension*, International Journal of Commutative Rings **1** No. 4 (2002), 195–201.
25. J. C. Rosales, P. A. García Sanchez, *Numerical Semigroups*, Springer (2009).
26. J. C. Rosales, P. A. García Sanchez, *On complete intersection affine semigroups*, Communications in Algebra **23** (1995), 5395–5412.
27. J. C. Rosales, P. A. García Sanchez, J. I. García-García, *k-factorized elements in telescopic numerical semigroups*, Arithmetical Properties of Commutative Rings and Monoids **241** (2005), 260–271.
28. K. Watanabe, *Some examples of one dimensional Gorenstein domains*, Nagoya Mathematical Journal **49** (1973), 101–109.
29. O. Zariski, *Le probléme des modules pour les branches planes*, Hermann, Paris, 1986.

(M. D'Anna) Università di Catania, Dipartimento di Matematica e Informatica, Viale A. Doria, 6, 95125 Catania, Italy
E-mail address: `mdanna@dmi.unict.it`

(V. Micale) Università di Catania, Dipartimento di Matematica e Informatica, Viale A. Doria, 6, 95125 Catania, Italy
E-mail address: `vmicale@dmi.unict.it`

(A. Sammartano) Department of Mathematics, Purdue University 150 N. University Street, West Lafayette, IN 47907, USA
E-mail address: `asammart@math.purdue.edu`